POISSON PROCESSES AND COMPOUND POISSON PROCESSES IN INSURANCE MANAGEMENT

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ABSTRACT

Some assumptions with respect to the number $\{N(t)\}_{t\geq 0}$ and the amount $\{X_i\}_{i=1}^{\infty}$ of damages are introduced in the paper. It will be assumed that the average of the number of damages is a Poisson process, which leads to a compound Poisson process $\{S(t)\}_{t\geq 0}$ for the total damages.

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1. POISSON PROCESS

Since the behaviour of the binomial distribution is $B(n,p)$, when $n\to\infty$ and $p\to 0$, thereby $E[X]=np=a$, where $a>0$ is a constant. On the basis of

the fact that

$$\lim_{n\to\infty}\left(q+p\exp(it)\right)^n=\lim_{n\to\infty}\left[1+\frac{a}{n}\left(\exp(it)-1\right)\right]^n=\exp\left\{a\left[\exp(it)-1\right]\right\},$$

and
Theorem 1[1]

A sequence of distribution functions $x\mapsto F_n(x)(n=1,2,...,)$ converges to the distribution function $x\mapsto F_n(x)$ for every $x\in\mathbb{R}$ for which F is a continuous function only then when the associated sequence $t\mapsto\varphi_n(t),(n=1,2,...)$ of characteristic functions converges to function $t\mapsto\varphi(t)$, which is continuous for $t=0$.

[1] Željko Pauše, Vjerojatnost, informacija, stohastički procesi, Školska knjiga, Zagreb, 1978, p. 139.

Then $t \mapsto \varphi(t)$ is a characteristic function that belongs to the distribution function $x \mapsto F(x)$.,

we conclude that in this case the binomial distribution $B(n,p)$ turns into a discrete probability distribution on the set $\{0,1,2,...\}$, to which there belong a characteristic function

$$\varphi(t) = \exp\{a[\exp(it)-1]\}$$

and a probability generating function $G(z) = \exp[a(z-1)] = \sum_{k=0}^{\infty} \frac{a^k}{k!} e^{-a} z^k$.

We have $$P(X = k) = p_k = \frac{a^k}{k!} e^{-a}, k = 0,1,...$$

The probability distribution defined by the previously mentioned expression is called a Poisson distribution with parameter a and it is denoted by $Po(a)$.

It is not difficult to conclude that the expectation and dispersion of a random variable X are Poisson distributed:

$$E[X] = a, D[X] = a,$$

and it can also be shown that the third and the fourth central moments of the random variable X are equal to the parameter $a (\mu_3 = \mu_4 = a)$.

One type of discrete-valued stochastic processes with independent stationary increments called a Poisson process is used for describing many practical phenomena.

If every $t > 0$:

$$P(X_t = k) = \frac{(bt)^k}{k!} \exp(-bt); k = 0,1,2,...,$$

where $b = E[X_1] > 0$, a stochastic process $\{X_t : t \in [0,\infty)\}$ is called a Poisson process.

Poisson law of distribution $P_0[b(t_2 - t_1)]$ also belongs to increment $X_{t_2} - X_{t_1} (t_2 > t_1 \geq 0)$, which is a random variable distributed as $X_{t_2 - t_1}$, so that

$$P\left(X_{t_2} - X_{t_1} = k\right) = \frac{\left[b\left(t_2 - t_1\right)\right]^k}{k!} \exp\left[-b\left(t_2 - t_1\right)\right], k = 0,1,\ldots$$ [2]

Poisson process is an example of a counting process[3]. We are interested here in the number of damages resulting from risks. Since damage is counted over time, the process of counting damages should satisfy the following conditions:

- $N(0)=0$, i.e. no damage in time 0,
- for every $t>0, N(t)$ must be an integer,
- for $s<t, N(s)<N(t)$, i.e. the number of damages over time is non-decreasing,
- for $s<t, N(t)-N(s)$ represents the number of damages occurring in the interval (s,t).

The process of the number of damages $\{N(t)\}_{t\geq 0}$ is defined as a Poisson process with parameter λ if the following conditions are satisfied:

a) $N(0)=0$ i $N(s)\leq N(t)$, for $s<t$,

b) $P\left(N(t+h)=r \middle| N(t)=r\right)=1-\lambda h+o(h)$

$P\left(N(t+h)=r+1 \middle| N(t)=r\right)=\lambda h+o(h)$

$P\left(N(t+h)>r+1 \middle| N(t)=r\right)=o(h)$

for $s<t$, the number of damages in the interval $(s,t]$ is independent of the number of damages by the moment s.

Condition (b) says that in a very short time interval of length h the only possible number of damages is equal to zero or one. Note that condition (b) also implies that the number of damages in the time interval of length h does not depend on when the time interval begins.

The reason why the process satisfying conditions (a) to (c) is called a Poisson process is that for a fixed value of t, the random variable $N(t)$ has a Poisson distribution with parameter λt. This is proved in the following way:

Let $p_n = P\left(N(t)=n\right)$. We will prove that

$$p_n = P\left(N(t)=n\right) \qquad (1)$$

[2] Željko Pauše, Vjerojatnost, informacija, stohastički procesi, Školska knjiga, Zagreb, 1978, p. 167.

[3] Nikola Sarapa, Teorija vjerojatnosti, Školska knjiga, Zagreb, 1987, pp. 382-387.

by deriving and solving a "differential-difference" equation[4].

For a fixed value $t>0$ and a small positive value h, we write

$$\begin{aligned} p_n(t+h) &= p_{n-1}(t)\left[\lambda h + o(h)\right] + p_n(t)\left[1-\lambda h + o(h)\right] + o(h) \\ &= \lambda h p_{n-1}(t) + \left[1-\lambda h\right] p_n(t) + o(h) \end{aligned}$$

Hence

$$p_n(t+h) - p_n(t) = \lambda h\left[p_{n-1}(t) - p_n(t)\right] + o(h) \qquad (2)$$

And this identity holds for $n = 1,2,3,...$

Now, if we divide (2) by h and if let h towards zero from the right-hand side, we obtain a differential-difference equation

$$\frac{d}{dt} p_n(t) = \lambda\left[p_{n-1}(t) - p_n(t)\right]. \qquad (3)$$

For $n=0$, an identical analysis yields

$$\frac{d}{dt} p_0(t) = -\lambda p_0(t) \qquad (4)$$

$p_n(t)$ is solved by introducing a probability function $G(s,t)$ defined by

$$G(s,t) = \sum_{n=0}^{\infty} s^n p_n(t),$$

such that

$$\frac{d}{dt} G(s,t) = \sum_{n=0}^{\infty} s^n p_n(t).$$

Let us now multiply (3) by s^n and sum over all values n in order to get

$$\sum_{n=1}^{\infty} s^n \frac{d}{dt} p_n(t) = \lambda \sum_{n=1}^{\infty} s^n p_{n-1}(t) - \lambda \sum_{n=1}^{\infty} s^n p_n(t).$$

If (4) is added to the above identity, we obtain

$$\sum_{n=0}^{\infty} s^n \frac{d}{dt} p_n(t) = \lambda \sum_{n=1}^{\infty} s^n p_{n-1}(t) - \lambda \sum_{n=0}^{\infty} s^n p_n(t),$$

which can be written as

4 Darko Veljan, Kombinatorika s teorijom grafova, Školska knjiga, Zagreb, 1989, p.220.

$$\frac{d}{dt}G(s,t)=\lambda sG(s,t)-\lambda G(s,t),$$

or equivalently

$$\frac{1}{G(s,t)}\frac{d}{dt}G(s,t)=\lambda(s-1). \qquad (5)$$

Since the left-hand side of (5) is equal to the derivative of $\log(s,t)$ at *t*, (5) can be integrated so that we obtain

$$\log G(s,t)=\lambda t(s-1)+c(s),$$

where $c(s)$ is some function of s. $c(s)$ can be identified if we note that for $t=0, p_0(t)=1$ and $p_n(t)=0, n=1,2,3,...$ Hence $G(s,0)=1$ and

$\log G(s,0)=0=c(s)$. Therefore $G(s,t)=e^{\lambda t(s-1)}$,

which is a probability generating function of the Poisson distribution[5] with parameter λt. Since there exists a one-to-one relationship between probability generating functions and distribution functions, it follows that the distribution $N(t)$ is a Poisson distribution with parameter λt. We will complete this study of the Poisson process by considering distribution of the time until the first damage and the time between damages.

Some random variable T_1 denotes the time of the first damage. For a fixed value t, if none of the damages occurred by the moment t, we have $T_1>t$. There follows

$$P(T_1>t)=P\big(N(T)=0\big)=e^{-\lambda t}$$

and

$$P(T_1\le t)=1-e^{-\lambda t}$$

so that T_1 has an exponential distribution with parameter λ.

For $n=2,3,...$, some random variable T_i denotes the time between the $(i-1)$-th and the $i-$th damage. Then

$$P\left(T_{n+1}>t\,\middle|\,\sum_{i=1}^{n}T_i=r\right)=P\left(\sum_{i=1}^{n+1}T_i>t+r\,\middle|\,\sum_{i=1}^{n}T_i=r\right)$$

5 Ivo Pavlić, Statistička teorija i primjena, Tehnička knjiga, Zagreb, 1970, pp. 79-83.

$$= P\left(N(t+r) = n \middle| N(r) = n\right)$$
$$= P\left(N(t+r) - N(r=0) \middle| N(r) = n\right).$$

According to condition (2),

$$P\left(N(t+r) - N(r) = 0 \middle| N(r) = n\right) = P\left(N(t+r) - N(r) = 0\right).$$

Finally,

$$P\left(N(t+r) - N(r) = 0\right) = P\left(N(t) = 0\right) = e^{-\lambda t},$$

since the number of damages in the time interval of length r does not depend on when the time interval begins. Therefore, the times between events also have the exponential distribution with parameter λ.

2. COMPOUND POISSON PROCESS

We will combine the Poisson process of the number of damages with distribution of the amount of damages and in this way we will obtain a compound Poisson process for the process of the total damages.

We will make the following three important assumptions:

- random variables $\{X_i\}_{i=1}^{\infty}$ are independent and equally distributed,
- random variables $\{X_i\}_{i=1}^{\infty}$ are independent of $N(t)$, for all $t \geq 0$,
- random process $\{N(t)\}_{t\geq 0}$ is a Poisson process whose parameter is denoted by λ.

It has been shown previously that the last assumption implies that for every $t \geq 0$ the random variable $N(t)$ has the Poisson distribution with parameter λt, such that

$$P\left[N(t) = k\right] = e^{-\lambda t}\frac{(\lambda t)^k}{k!}, \text{ for } k = 0,1,2,...$$

With these assumptions, the total damage average $\{S(t)\}_{t\geq 0}$ is called a compound Poisson process with Poisson parameter λ.

It can be easily seen that if $\{S(t)\}_{t\geq 0}$ is a compound Poisson process with Poisson parameter λ, then for a fixed value of $t \geq 0$, $S(t)$ has a compound Poisson distribution with Poisson parameter λt.

By making a change from a process to a distribution “Poisson parameter λ“ becomes “Poisson parameter λt“.

A common distribution function of X_i will be denoted by $F(x)$, and we will assume $F(0)=0$, so that all damages are positive.

A probability density function of X_i, if it exists, will be denoted by $f(x)$, the k-th moment about zero of X_i, if it exists, will be denoted by m_k, so that

$$m_k = E\left[X_i^k\right], \text{ for } k = 0,1,2,...$$

Let us mention some properties of sample moments.

Let X be a property with the distribution function F, for which it holds $E(X)=m$, $\mathrm{E}\left(X^{-k}\right)=m_k$. Consider a simple random sample $(\mathrm{X}_1,\mathrm{X}_2,...,\mathrm{X}_n)$ of size n chosen from variable F.

The sample mean is a statistic that represents the arithmetic mean of the components of samples;

$\overline{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k$, whereby $\overline{X}_n$ represents a sample equivalent to mathematical expectation $E(X)$ by a theoretical distribution of property X.

Note that a simple random sample of size n from the population in which property X has a distribution function $F(x)$, is a random vector $\left(X_{1,}X_2,...,X_n\right)$, whereby $X_{1,}X_2,...,X_n$ are independent random values with the same distribution function $F(x)$ as property X.

The distribution function of a random vector $(X_1,X_2,...,X_n)$ is

$$F(x_1,x_2,...,x_n) = P\{X_1 \le x_1, X_2 \le x_2,..., X_n \le x_n\} = P\{X_1 \le x_1\}P\{X_2 \le x_2\}...P\{X_n \le x_n\} = \prod_{k=1}^{n} F(x_k)$$

Furthermore, an ordinary sample moment of order k is a statistic $A_{nk} = \frac{1}{n}\sum_{i=1}^{n} x_i^k$ and it represents a sample equivalent to a theoretical ordinary moment of order k defined by $m_k = E\left(\mathrm{X}^k\right)$.

It is not difficult to show that $E\left(\overline{X}_n\right)=m$ and $E(\mathrm{A}_{nk})=m_k$.

Indeed, $E\left(\overline{X}_n\right) = E\left(\frac{1}{n}\sum_{k=1}^{n} X_k\right) = \frac{1}{n}\sum_{k=1}^{n} E(X_k) = \frac{1}{n}\cdot n\cdot m = m,$

$$E\left(\mathrm{A}_{nk}\right)=E\left(\frac{1}{n}\sum_{i=1}^{n}X_i^k\right)=\frac{1}{n}\sum_{i=1}^{n}E\left(X_i^k\right)=\frac{1}{n}\cdot n\cdot m_k=m_k\ .$$

Whenever there exists a common moment generating function[6] of X_i, its value at point r will be denoted by $M_x(r)$.

Since for a fixed value of *t*, $S(t)$ has a compound Poisson distribution, it can be easily shown that process $\{S(t)\}_{t\geq 0}$ has expectation $\lambda t m_1$, variance $\lambda t m_2$ and moment generating function $M_s(r)$, where $M_s(r)=e^{\lambda t(M_x(r)-1)}$.

Conclusion

In addition to some assumptions with respect to the number and the amount of damages, the paper shows that for a fixed value of t $N(t)$ has a Poisson distribution with parameter λt. When a change from a process to a distribution is made, "Poisson parameter λ" becomes "Poisson parameter λt". Distributions of the time until the first damage and the time between damages are considered. It is shown that the times between events have the exponential distribution with parameter λ.

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